WEB OF SYNERGY:

International Interdisciplinary Research Journal

*Volume 2 Issue 6, Year 2023 ISSN: 2835-3013*

*https://univerpubl.com/index.php/synergy*

# Navier-Stokes Equations for a Viscous Incompressible Fluid

**Dilafruz Nurjabova**

Tashkent University of Information Technologies, "Multimedia Technologies", 180118, Tashkent, Uzbekistan, dilyaranur1986@gmail.com

**Abdullaev Ravshan Narszullaevich**

Tashkent University of Information Technologies"Software Engineering", 180118, Tashkent, Uzbekistan

**Article Information**

**Received:** April 27, 2023

**Accepted:** May 27, 2023

**Published:** June 28, 2023

**Keywords:** *viscosity parameter, navier-stokes equations, fluid, unbounded domain.*

**ABSTRACT**

*The solution of the Navier-Stokes equations for a viscous incompressible fluid in an unbounded domain has significant applications in the field of medicine, in particular, for modeling blood flow in the heart and blood vessels. In this context, the annotation can be described as follows: This work is devoted to the study of solutions of the Navier-Stokes equations for a viscous incompressible fluid in an unlimited region with application to the modeling of blood flow in the heart and blood vessels. The main attention is paid to the development of mathematical models and numerical methods for solving this problem, as well as the analysis of the results obtained.*

## I.INTRODUCTION

The state of a moving fluid is determined by setting five values: three components of velocity *V(x; y; z; t)* pressure *p*(*x; y; z; t*) and density _(*x; y; z; t*) In fluid mechanics, its molecular structure is not considered, it is assumed that the fluid fills the space entirely, instead of the fluid itself, its model is studied, a fictitious continuous medium with the property of continuity. This approach simplifies the researching, all mechanical and hemodynamics characteristics of the liquid medium (velocity, pressure, density) are assumed to be continuous and differentiable.

The equations of motion of a viscous incompressible fluid (Navier-Stokes equations) in projections on the coordinate axis by velocity components have the form [1]

$$\frac{\partial \upsilon_x}{\partial t} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} - \upsilon_x \frac{\partial \upsilon_x}{\partial x} - \upsilon_y \frac{\partial \upsilon_x}{\partial y} - \upsilon_z \frac{\partial \upsilon_x}{\partial z} + \nu\left(\frac{\partial^2 \upsilon_x}{\partial x^2} + \frac{\partial^2 \upsilon_x}{\partial y^2} + \frac{\partial^2 \upsilon_x}{\partial z^2}\right), \tag{1}$$

$$\frac{\partial \upsilon_y}{\partial t} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} - \upsilon_x \frac{\partial \upsilon_y}{\partial x} - \upsilon_y \frac{\partial \upsilon_y}{\partial y} - \upsilon_z \frac{\partial \upsilon_y}{\partial z} + \nu\left(\frac{\partial^2 \upsilon_y}{\partial x^2} + \frac{\partial^2 \upsilon_y}{\partial y^2} + \frac{\partial^2 \upsilon_y}{\partial z^2}\right), \tag{2}$$

$$\frac{\partial \upsilon_z}{\partial t}=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}-\upsilon_x\frac{\partial \upsilon_z}{\partial x}-\upsilon_y\frac{\partial \upsilon_z}{\partial y}-\upsilon_z\frac{\partial \upsilon_z}{\partial z}+\nu\left(\frac{\partial^2 \upsilon_z}{\partial x^2}+\frac{\partial^2 \upsilon_z}{\partial y^2}+\frac{\partial^2 \upsilon_z}{\partial z^2}\right), \quad (3)$$

where $\mathbf{V}(x,y,z,t)=\upsilon_x(x,y,z,t)\cdot\mathbf{i}+\upsilon_y(x,y,z,t)\cdot\mathbf{j}+\upsilon_z(x,y,z,t)\cdot\mathbf{k},$; values of x; y; z; t – are called Euler variables, $\mathbf{F}(x,y,z,t)=F_x(x,y,z,t)\mathbf{i}+F_y(x,y,z,t)\mathbf{j}+F_z(x,y,z,t)\mathbf{k}$ -is the intensity of the field of mass forces *grad* $p=\nabla p=\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}\frac{\partial p}{\partial z}\mathbf{k},\ p=p(x,y,z),$

$\nabla$ – operator "nabla" , $\rho$ – density of the liquid, υ– kinematic viscosity of the liquid,
*i, j, k* – orts.

The continuity equation for an incompressible fluid $\frac{dy}{dt}=0$:

$$div\mathbf{V}=\frac{\partial \upsilon_x}{\partial x}+\frac{\partial \upsilon_y}{\partial y}+\frac{\partial \upsilon_z}{\partial z}=0,\ \ \forall(x,y,z),\forall t \quad (4)$$

The main task of hydrodynamics is to find the following functions of coordinates and time: $\upsilon_x=f_1(x,y,z,t)$

$\upsilon_y=f_2(x,y,z,t),\upsilon_z=f_3(x,y,z,t),\ p=f_4(x,y,z,t)$ under the given initial conditions $(\rho=const>0)$. (5)

$\upsilon_x|_{t=0}=f_1(x,y,z,0),\upsilon_y|_{t=0}=f_2(x,y,z,0),\upsilon_z|_{t=0}=f_3(x,y,z,0)$ under the given initial conditions.

## II. MAIN PART (6)

The equations of motion of a viscous incompressible fluid (1)-(4), tested in practice, adequately reflect the physical phenomenon in liquids and are a correct mathematical model. Therefore, the equations of motion (1)-(3) and continuity (4) are sufficient to solve the main problem of hydrodynamics when $v_x(x;\ y;\ z;\ t)$; $v_y(x;\ y;\ z;\ t)$;$v_z(x;\ y;\ z;\ t)$ – continuously differentiable functions with respect to t and twice continuously differentiable functions with respect to variables *x; y; z;* in the domain $(x,y,z)\in\Omega=R^3,t\in T=\{t\in R^1/t>0\}$.

$$\upsilon_x(x,y,z,t)\in C^{2,2,2,1}_{x,y,z,t}(\Omega\times T),\upsilon_y(x,y,z,t)\in C^{2,2,2,1}_{x,y,z,t}(\Omega\times T),\ \upsilon_z(x,y,z,t)\in C^{2,2,2,1}_{x,y,z,t}(\Omega\times T)$$

We assume that $F_x(x,y,z,t),F_y(x,y,z,t),\ F_z(x,y,z,t)$ are given continuous functions in the domain of $\Omega\times T$.

$F_x(x,y,z,t)\in C_{x,y,z,t}(\Omega\times T),F_y(x,y,z,t)\in C_{x,y,z,t}(\Omega\times T),F_z(x,y,z,t)\in C_{x,y,z,t}(\Omega\times T)$ and functions $p_x(x,y,z,t)=\frac{\partial\rho}{\partial x}\in C_{x,y,z,t}$, $p_y(x,y,z,t)=\frac{\partial\rho}{\partial y}\in C_{x,y,z,t}$, $p_z(x,y,z,t)=\frac{\partial\rho}{\partial z}\in C_{x,y,z,t}$.

In the classical formulation, the initial problem for the Navier-Stokes equations for a viscous incompressible fluid in an unbounded domain $\Omega\times T$ has the form: find functions, $\upsilon_x(x,y,z,t):\Omega\times T\to R^1,\upsilon_y(x,y,z,t):\Omega\times T\to R^1,\ \upsilon_z(x,y,z,t):\Omega\times T\to R^1$

such that they satisfy equations (1)-(3) in Ω×T and the continuity equation (4) under given initial conditions (6), where $f_i(x,y,z,0)\in C(\Omega),\left|f_i(x,y,z,0)\right|\le c_i,c_i=const>0\ ,i=1,2,3$.

The proposed method for solving this problem is obtained on the basis of the author's works published in [5-7]. For simplicity of presentation of the results obtained, a regular solution of the Navier-Stokes equation is given below.

Suppose that solutions of system (1)-(4) with initial condition (6) are known.

Then

$$\frac{\partial \upsilon_x}{\partial t} = \nu\left(\frac{\partial^2 \upsilon_x}{\partial x^2} + \frac{\partial^2 \upsilon_x}{\partial y^2} + \frac{\partial^2 \upsilon_x}{\partial z^2}\right) + \omega(x, y, z, t)$$

$$\frac{\partial \upsilon_y}{\partial t} = \nu\left(\frac{\partial^2 \upsilon_y}{\partial x^2} + \frac{\partial^2 \upsilon_y}{\partial y^2} + \frac{\partial^2 \upsilon_y}{\partial z^2}\right) + \mu(x, y, z, t)$$

$$\frac{\partial \upsilon_z}{\partial t} = \nu\left(\frac{\partial^2 \upsilon_z}{\partial x^2} + \frac{\partial^2 \upsilon_z}{\partial y^2} + \frac{\partial^2 \upsilon_z}{\partial z^2}\right) + \theta(x, y, z, t)$$

Where

$$\omega(x, y, z, t) = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} - \upsilon_x \frac{\partial \upsilon_x}{\partial x} - \upsilon_y \frac{\partial \upsilon_x}{\partial y} - \upsilon_z \frac{\partial \upsilon_x}{\partial z}$$

$$\mu(x, y, z, t) = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} - \upsilon_x \frac{\partial \upsilon_y}{\partial x} - \upsilon_y \frac{\partial \upsilon_y}{\partial y} - \upsilon_z \frac{\partial \upsilon_y}{\partial z}$$

$$\theta(x, y, z, t) = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} - \upsilon_x \frac{\partial \upsilon_z}{\partial x} - \upsilon_y \frac{\partial \upsilon_z}{\partial y} - \upsilon_z \frac{\partial \upsilon_z}{\partial z}$$

in front of everything $(x, y, z) \in \Omega = R^3, t \in T$.

Let the function

$U(x, y, z, t) \in C^{2,2,2,1}(\Omega \times T), V(x, y, z, t) \in C^{2,2,2,1}(\Omega \times T), W(x, y, z, t) \in C^{2,2,2,1}(\Omega \times T)$ – solutions of the following equations

$$\frac{\partial U}{\partial t} = \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) + \omega_1(x, y, z, t)$$

$$\frac{\partial V}{\partial t} = \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) + \mu_1(x, y, z, t)$$

$$\frac{\partial W}{\partial t} = \nu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right) + \theta_1(x, y, z, t)$$

with initial conditions

$U|_{t=0} = f_1(x, y, z, 0), V|_{t=0} = f_2(x, y, z, 0), W|_{t=0} = f_3(x, y, z, 0),$

in the area of $(x, y, z) \in \Omega = R^3, t \in T$

In here $\omega_1(x, y, z, t) \in C_{x,y,z,t}(\Omega \times T), \mu_1(x, y, z, t) \in C_{x,y,z,t}(\Omega \times T), \theta_1(x, y, z, t) \in C_{x,y,z,t}(\Omega \times T)$

– unknown bounded absolutely integral continuous functions – unknown bounded absolutely integral continuous functions. In particular if

$\omega_1(x,y,z,t)=\omega(x,y,z,t)$, $\mu_1(x,y,z,t)=\mu(x,y,z,t)$, $\theta_1(x,y,z,t)=\theta(x,y,z,t)$ else $U(x,y,z,t)=\upsilon_x(x,y,z,t)$, $V(x,y,z,t)=\upsilon_y(x,y,z,t)$, $W(x,y,z,t)=\upsilon_z(x,y,z,t)$ in $(x,y,z)\in\Omega, t\in T$.

Systems of parabolic equations (13)-(15) with initial conditions (16) have solutions [8, 9]:

$$U(x,y,z,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\left(2\sqrt{\nu\pi t}\right)^3}f_1(\xi,\eta,\zeta)e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu t}}d\xi d\eta d\zeta+$$
$$+\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\omega_1(\xi,\eta,\zeta,\tau)}{\left(2\sqrt{\pi\nu(t-\tau)}\right)^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu t}}d\xi d\eta d\zeta d\tau \tag{17}$$

$$V(x,y,z,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f_2(\xi,\eta,\zeta)}{\left(2\sqrt{\nu\pi t}\right)^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu t}}d\xi d\eta d\zeta+$$
$$+\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mu_1(\xi,\eta,\zeta,\tau)}{\left(2\sqrt{\pi\nu(t-\tau)}\right)^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu(t-\tau)}}d\xi d\eta d\zeta d\tau \tag{18}$$

$$W(x,y,z,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f_3(\xi,\eta,\zeta)}{\left(2\sqrt{\nu\pi t}\right)^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu t}}d\xi d\eta d\zeta+$$
$$+\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\theta_1(\xi,\eta,\zeta,\tau)}{\left(2\sqrt{\pi\nu(t-\tau)}\right)^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu(t-\tau)}}d\xi d\eta d\zeta d\tau \tag{19}$$

Introducing notation $G(x,y,z,t)=\frac{1}{(2\sqrt{\pi\nu t})^3}e^{-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4\nu t}}$, $d\sigma=d\xi d\eta d\zeta$

solutions (17)-(19) are written as

$$U(x,y,z,t)=\int_\Omega G(x,y,z,t,\sigma)f_1(\sigma)d\sigma+\int_0^t\int_\Omega G(x,y,z,t-\tau,\sigma)\omega_1(\sigma,\tau)d\sigma d\tau \tag{20}$$

$$V(x,y,z,t)=\int_\Omega G(x,y,z,t,\sigma)f_2(\sigma)d\sigma+\int_0^t\int_\Omega G(x,y,z,t-\tau,\sigma)\mu_1(\sigma,\tau)d\sigma d\tau \tag{21}$$

$$W(x,y,z,t)=\int_\Omega G(x,y,z,t,\sigma)f_3(\sigma)d\sigma+\int_0^t\int_\Omega G(x,y,z,t-\tau,\sigma)\theta_1(\sigma,\tau)d\sigma d\tau \tag{22}$$

Where $\Omega=R^3, \sigma=(\xi,\eta,\zeta)$ we note that $\omega_1(\sigma,\tau), \mu_1(\sigma,\tau), \theta_1(\sigma,\tau)-$ unknown bounded absolutely integral continuous functions in $\tau\in\{0\le\tau<t, t\in T\}=T_1$

$\omega_1(\sigma,\tau)$, $\mu_1(\sigma,\tau)$, $\theta_1(\sigma,\tau)$, $\sigma\in\Omega$, $\tau\in T_1$

The Navier-Stokes equations (1)-(3) for a viscous incompressible fluid under any initial conditions (6) have solutions satisfying the continuity equation (4) in an unbounded domain if and only if the function $\Gamma(x,y,z,t)>0$ for any $(x,y,z)\in\Omega=R^3$; $t\in\mathrm{T}$; where Γ(x; y; z; t) is determined by the formula

$$\int_0^t\int_\Omega P(x,y,z,t-\tau,\sigma)\Lambda(\sigma,\tau)d\sigma d\tau=R(x,y,z,t)$$

Solutions $\upsilon_x(x,y,z,t)=S(x,y,z,t,\chi)$, $\upsilon_y(x,y,z,t)=\mathrm{T}(x,y,z,t,\chi)$, $\upsilon_z(x,y,z,t)=Q(x,y,z,t,\chi)$, $\upsilon_x(x,y,z,t)=S_0(x,y,z,t,\chi)$, $\upsilon_y(x,y,z,t)=\mathrm{T}_0(x,y,z,t,\chi)$, $\upsilon_z(x,y,z,t)=Q_0(x,y,z,t,\chi)$, in general, are not unique and are determined by formulas, respectively.

**III. DISCUSSION AND SOLUTION**

Solution
$\upsilon_x(x,y,z,t)=S(x,y,z,t,\chi)$, $\upsilon_y(x,y,z,t)=\mathrm{T}(x,y,z,t,\chi)$, $\upsilon_z(x,y,z,t)=Q(x,y,z,t,\chi)$, $\upsilon_x(x,y,z,t)=S_0(x,y,z,t,\chi)$, $\upsilon_y(x,y,z,t)=\mathrm{T}_0(x,y,z,t,\chi)$, $\upsilon_z(x,y,z,t)=Q_0(x,y,z,t,\chi)$, the only one if

$$\int_0^t\int_\Omega \mathrm{P}_1(x,y,z,t-\tau,\sigma)\chi(\sigma,\tau)d\sigma d\tau\equiv 0,\quad \int_0^t\int_\Omega \mathrm{P}_2(x,y,z,t-\tau,\sigma)\chi(\sigma,\tau)d\sigma d\tau\equiv 0$$

$$\int_0^t\int_\Omega \mathrm{P}_3(x,y,z,t,t-\tau,\sigma)\chi(\sigma,\tau)d\sigma d\tau\equiv 0,\quad \forall(x,y,z)\in\Omega=R^3,\ \ t\in\mathrm{T},$$

$\mathrm{P}_1,\mathrm{P}_2,\mathrm{P}_3$ where is

$$S_0(x,y,z,t)=\alpha_1(x,y,z,t),$$

$$\mathrm{T}_0(x,y,z,t)=\alpha_2(x,y,z,t),$$

$Q_0(x,y,z,t)=\alpha_3(x,y,z,t)$, defined by formulas

$$\pi(x,y,z,t-\tau,\sigma)=\begin{pmatrix}\mathrm{P}_1(x,y,z,t-\tau,\sigma)\\ \mathrm{P}_2(x,y,z,t-\tau,\sigma)\\ \mathrm{P}_3(x,y,z,t-\tau,\sigma)\end{pmatrix}$$

condition is written in the form of an integral equation with respect to $\chi(\sigma,\tau)$:

$$\int_0^t\int_\Omega \pi(x,y,z,t-\tau,\sigma)\chi(\sigma,\tau)d\sigma d\tau=0$$

**V. CONCLUSION**

It is known that the Navier-Stokes equations for a viscous incompressible fluid under given initial conditions, as well as with the continuity condition, are among the most fundamental equations in hydrodynamics. An important issue is the existence and uniqueness of solutions to these equations with respect to the components of the fluid velocity. For this question, there is a theorem on the existence and uniqueness of solutions for the Navier-Stokes equations.

**REFERENCES**

1. *Landau L.D., Lifshits E.M.* Teoreticheskaya fizika. Gidrodinamika. – M.: Nauka, 1950. –T. 4 – 735 p.
2. *Ladyzhenskaya O.A.* Matematicheskie voprosy dinamiki vyazkoi zhidkosti neszhimaemoizhidkosti // – M.: Nauka, 1970. – 435 p.
3. *Temam R.* Uravneniya Nav'e-Stoksa. Teoriya i chislennyi analiz // – M.: Mir, 1981. –386 p.
4. *Fursikov A.V.* Optimal'noe upravlenie raspredelennymi sistemami. Teoriya i prilozheniya // – Novosibirsk: Nauchnaya kniga, 1999. – 352 p.
5. *Aisagaliev S.A., Belogurov A.P., Sevryugin I.V.* K resheniyu integral'nogo uravneniyaFredgol'ma pervogo roda dlya funktsii neskol'kih peremennyh // Vestnik KazNU, ser.mat., meh., inf. – 2011. –№ 1 (68). – P. 21-30.
6. *Aisagaliev S.A., Belogurov A.P., Sevryugin I.V.* Upravlenie teplovymi protsessami //Vestnik KazNU, ser. mat., meh., inf. – 2012, – № 1(72). – P. 14-26. (Rabota vypolnena pri podderzhke grantovogo finansirovaniya nauchno-tehnicheskih programm Komiteta nauki MON RK, grant № 0696 / GF, 2012 - 2014 gg.)
7. *Aisagaliev S.A., Belogurov A.P.* Upravlyaemost' i bystrodeistvie protsessa, opisyvaemogoparabolicheskim uravneniem s ogranichennym upravleniem // Sibirskii matematicheskiizhurnal. – 2012, – t. 53, № 1. – P. 20-37.
8. *Tihonov A.N., Samarskii A.A.* Uravneniya matematicheskoi fiziki // – M.: Nauka, 2004.– 798 p.
9. *Smirnov V.I.* Kurs vysshei matematiki. Tom vtoroi // – M.: Nauka, 1974. – 656 p.
10. Ciavolino, E., Lagetto, G., Montinari, A. et al. Customer satisfaction and service domains: a further development of PROSERV. Qual Quant 54, 1429–1444 (2020). https://doi.org/10.1007/s11135-019-00888-4